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Research into Practice

Mathematical Practice

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Al Cuoco, Author

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There is a practice of mathematics, just as there is a practice of medicine or a practice of teaching. There's been a great deal of recent discussion about "mathematical practice" or even "the practices," initiated in part by the Standards for Mathematical Practice that are part of the Common Core State Standards for Mathematics (CCSS-M) and the Process Standards from the National Council for Teachers of Mathematics (NCTM). Standards exemplify the ways of work employed by proficient users of mathematics—they are pillars supporting a whole practice rather than a taxonomy that covers the whole field.

Lists of standards are useful, but they are meant to accentuate a much larger style of work and an interconnected web of mathematical habits of mind used by mathematicians in their own work. Hyman Bass, an eminent American mathematician, puts it this way:

> It will be helpful to name and (at least partially) specify some of the things—practices, dispositions, sensibilities, habits of mind—entailed in doing mathematics . . . . These are things that mathematicians typically do when they do mathematics. At the same time, most of these things, suitably interpreted or adapted, could apply usefully to elementary mathematics no less than to research.[1]

The implication of Bass' quote—that there's a role in precollege mathematics for learning to do mathematics in ways similar to the styles of work employed by mathematics professionals—has been gaining traction for over a decade. It's now widely accepted that a modern high school program contains opportunities for students to both learn mathematics as an established body of knowledge and develop the ways of thinking that are indigenous to the discipline.

This essay gives some examples of how a focus on mathematical practice can help high school teachers bring both parsimony and mathematical coherence to the topics they teach.

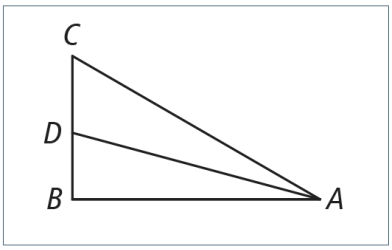
Let me illustrate with four examples. Each of these examples starts with a rather mundane task, and each of these tasks can be approached with many different methods. I want to illustrate specific methods that exemplify mathematical practice. The last example shows how various mathematical habits of mind come together over one investigation.

[1] Bass, Hyman (2011). *"Vignette of Doing Mathematics: A Meta-cognitive Tour of the Production of Some Elementary Mathematics,"* The Mathematics Enthusiast: Vol. 8: No. 1, Article 2. Available at: http://scholarworks.umt.edu/tme/vol8/iss1/2

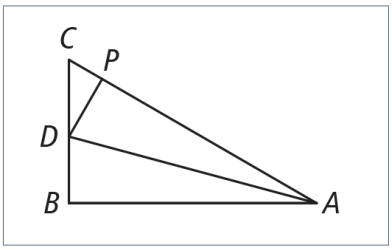
Example 1: Perseverance

This is a recount of an actual investigation conducted by a group of high school teachers in 2011. There are several morals to the story; one is discussed at the end.

The problem was to find an exact value for sin 15°. After some time, the first solution came from Amato who took a 30-60-90 triangle of hypotenuse length 2 and bisected the 30° angle:



Every trigonometry teacher has seen this: the assumption is that the angle bisector bisects the opposite leg. From this, you get a value for sin 15°. Several people pointed out right away that the assumption wasn't correct (and they verified this in dynamic geometry software). But rather than simply calling this a mistake, Sarah got up and said that she saw something. She drew an altitude from D to AC and noted that this formed $\triangle ADP$ that's congruent to $\triangle ADB$ with a little 30-60-90 $\triangle DCP$ leftover.



> "Standards exemplify the ways of work employed by proficient users of mathematics—they are pillars supporting a whole practice rather than a taxonomy that covers the whole field."

Several students jumped on this idea, arguing along these lines:

Since $AC = 2$ and $AP = AB = \sqrt{3}$, $PC = 2 - \sqrt{3}$

Since $\triangle DCP$ is 30-60-90, $CD = 2$, $PC = 4 - 2\sqrt{3}$.

But $BC = 1$, so $BD = 1 - CD = 2\sqrt{3} - 3$, and $DP = 2\sqrt{3} - 3$, too.

The Pythagorean theorem makes $AD = \sqrt{24 - 12\sqrt{3}}$

Hence $\sin 15° = \dfrac{2\sqrt{3} - 3}{\sqrt{24 - 12\sqrt{3}}}$

This worked out numerically the same as the calculator approximation, but the feeling was that to see some structure that might lead to a generalization, this could be simplified considerably.

So, groups started working on this. After a few minutes, Alicia worked out a board full of algebraic simplifications, stopping at each step to make sure that everyone understood. She ended up with

$\sin 15° = \dfrac{\sqrt{2 - \sqrt{3}}}{2}$

This again agreed numerically, and everyone thought it was a great simplification.

Then Kevin entered sin15° into his calculator and it produced $\dfrac{\sqrt{6} - \sqrt{2}}{4}$

Were these two things the same? Pat suggested squaring both, and sure enough, they were.

> "Alicia's work was a real tour de force, the kind of old fashioned algebraic calculations that many teachers love."

It was time to go home, but the group felt sure that this idea could be used to get the general half-angle formulas. And, for the next meeting, they decided to investigate how and when one can rewrite $\sqrt{a + \sqrt{b}}$ as the sum of two square roots of rational expressions in a and b.

These results certainly are not new or profound. It is the nature of the work itself—abstracting from numericals, using special cases to inspire generalizations, salvaging false starts—that makes it so faithful to real mathematical practice.

MORAL: Perseverance has many faces. One of them is that an incorrect conjecture often contains the germ of a good idea. Rather than abandoning a false start, mathematicians often dig into what went wrong, sometimes for long periods of time, seeing if the basic idea can be repaired or used in some other way to come to understanding. It often does. And the repair itself can often be used to launch new investigations—in this case, an investigation into equivalent algebraic expressions. This aspect of perseverance is extremely motivating for learners—the idea that not giving up sometimes leads one from incorrect assumptions to valid results can bootstrap the very practice of "sticking with it."

Example 2: Abstraction

Here's the kind of problem that often stumps students:
A music store offers piano lessons at a discount for customers buying new pianos. The costs for lessons and a one-time fee (for music books, CDs, software,) are shown in the advertisement.

How much is the one-time fee?
Every teacher knows that the obstacle here is coming up with the equation that models the situation—solving it is another (often easier) matter. Here's a method that builds on students' ability to solve similar problems in middle school: Take a guess. This is not to guess-and-check, getting closer to the answer each time. We want an equation, not a number. Of course, students need to understand that the cost per lesson doesn't depend on the number of lessons taken.

Take a guess, say $20.

Check it: (*)

(*) $\frac{300 - 20}{6} = 46\frac{2}{3}$

(*) $\frac{480 - 20}{12} = 38\frac{1}{3}$

(*) $46\frac{2}{3} \neq 38\frac{1}{3}$

That wasn't right, but that's okay—just keep track of your steps.

Take another guess, say $70, and check it:
$\frac{300 - 70}{6} = \frac{480 - 70}{12}$

Keep it up, until you get a "guess checker".
$\frac{300 - \text{guess}}{6} = \frac{480 - \text{guess}}{12}$

The equation is $\frac{300 - x}{6} = \frac{480 - x}{12}$

This method for finding an equation that models a situation builds on students' ability to do numerical calculations that lie behind pre-algebra problems. The idea is to check enough guesses so that you get the regularity of the calculations needed to check the guesses; a generic "guess checker" is the desired equation. So, it involves:

1. Working through several specific examples, concentrating on the "rhythm" of their calculations, and then
2. Expressing this regularity in precise mathematical language.

The method captures a very common habit that is useful throughout algebra: carry out several concrete examples of a process that you don't quite "have in your head" in order to find regularity and to build a generic algorithm that describes every instance of the calculation.

> "The underlying method has roots in the genetic epistemology of Jean Piaget[2]. It's a habit of mind that mathematicians use throughout their work: abstracting regularity from repeated calculations."

[2] Piaget, Jean (1977). *The Development of Thought. Equilibration of Cognitive Structures.* New York: Viking

MORAL: "Try it with numbers" is a slogan used by many teachers as a way to help students get started on a problem when they don't quite see how to model it with an algebraic expression. This advice can be used to help students build the actual expression by learning to orchestrate their numerical examples in a way that highlights the operations rather than the inputs to those operations. And the habit is useful all over mathematics, from building Cartesian equations that characterize geometric objects to building functions that model financial phenomena.

Example 3: Structure in Expressions

Factoring has gotten a bad name in recent years. But polynomial algebra sits at the historical core of algebra, and it plays a central role in current research.

Some factoring methods that can be introduced in elementary algebra are extensible, preview important ideas, and give students a chance to develop the habits of reasoning about calculations in algebraic structures and transforming a calculation in an invertible way to one that is easier to carry out.

Because

$$(x + r)(x + s) = x^2 + (r + s)x + rs,$$

factoring a quadratic polynomial, $x^2 + bx + c$ (with leading coefficient 1) amounts to finding numbers r and s such that

$$r + s = b \text{ and } rs = c$$

This "sum-product" approach is fairly tractable to beginning students. To factor $x^2 + 17x + 72$, students look for two numbers that add to 17 and multiply to 72, so that

$$x^2 + 17 + 72 = (x + 9)(x + 8)$$

Many students who can factor monic (leading coefficient of 1) quadratics by the sum-product approach often have much more difficulty when the leading coefficient is not 1. Here again, there are general purpose methods that live on beyond their utility for developing this particular skill. One such method starts with the observation that $4x^2 + 36x + 45$ can more easily be factored if one "chunks" the terms and writes it as

$$(2x)^2 + 18(2x) + 45$$

One can think of this as a "quadratic in $2x$," thinking of $2x$ as the variable. One can even replace $2x$ by some symbol, say z, and write the quadratic as

$$z^2 + 18z + 45$$

This factors by the sum-product method:

$$(z + 15)(z + 3)$$

Replacing z by $2x$ gives the factorization of the original quadratic.

The coefficients in this example were especially suited for this technique. What if one is faced with something like

$$6x^2 + 11x - 10?$$

One can reason like this:

Multiply the polynomial by 6 to make the leading coefficient a perfect square, remembering that we have to divide by 6 at some point to get back to where we started.

$$6(6x^2 + 11x - 10) = 36x^2 + 11 \bullet 6x - 60$$

This is a quadratic in 6x; let z = 6x, so the right-hand side becomes monic

$$z^2 + 11z - 60$$

This factors by the sum-product method

$$(z + 15)(z - 4)$$

But z = 6x, so we have

$$(6x + 15)(6x - 4)$$

Factor out common factors—3 from the first binomial and 2 from the second—producing

$$6(2x + 5)(3x - 2)$$

Dividing by 6 gives the factorization of the original polynomial.

Notice that the insight here did not come from abstracting regularity from numerical examples; rather it came from seeing a certain structure in the quadratic polynomial. "Seeing structure in expressions" is often described as looking for hidden meaning.

MORAL: This "scaling method" is a general-purpose tool that has applications all over algebra and calculus. It amounts to changing the variable in order to hide complexity. It can be used, for example, to transform a polynomial of any degree to one whose leading coefficient is 1. More generally, chunking—treating a piece of an expression as a single object—is another useful form of abstraction. For example, treating $x - 2$ as a single chunk in the expression

$$5 - (x - 2)^2$$

tells you that the maximum value for this expression (for all real values of x) is 5 (and that this occurs when $x = 2$).

> "Seeing structure in expressions is often described as looking for hidden meaning."

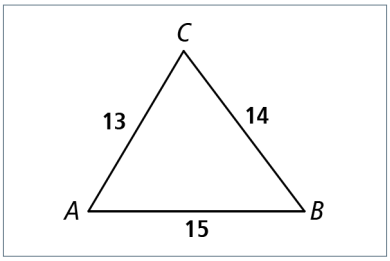
Example 4: Using Mathematical Practice

A triangle is determined by the lengths of its sides—this is the SSS theorem in geometry. Hence, one should be able to figure out the area of a triangle from its three side lengths. There is a formula for this, called Heron's formula, but it often seems mysterious to students.

Here's one approach to the formula that makes use of several aspects of mathematical practice in one task.

First of all, if I'm in the habit of abstracting from numericals, I'll want to find the area of a specific triangle and see if I can do it in a way that doesn't depend on the actual numbers. So, I start with the famous 13-14-15 triangle:

I know that the area is ½ times the base times the height, so I draw in a height and label some new pieces:



Even here, if I'm attuned to looking for structure, I see a method lurking that will work for any numerical example. Plowing on, I set up two equations, using the Pythagorean theorem:

$x^2 + h^2 = 169$

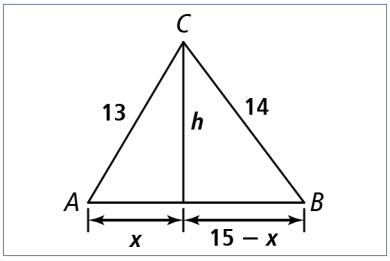
$(15 - x) + h^2 = 196$

Expand the second equation:

$225 - 30x + x^2 + h^2 = 196$

Ah, but $x^2 + h^2 = 169$ from the first equation, so

$225 - 30x + 169 = 196$



From here, you can find x, then h, and then finally the area.

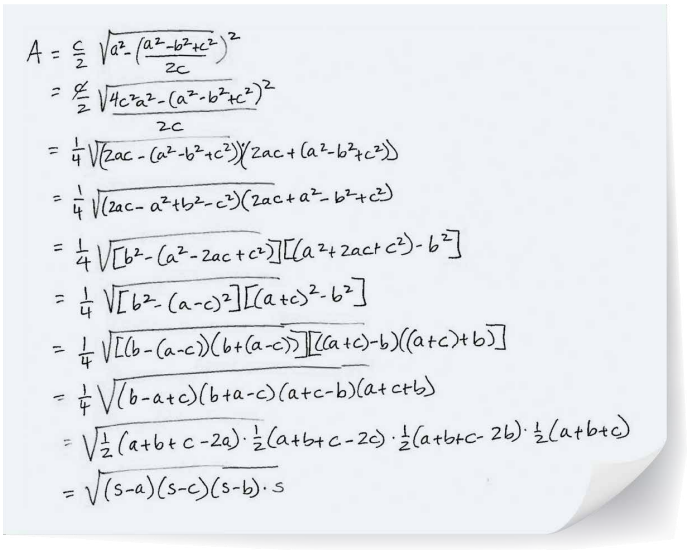
If I were doing this for real, I'd try it with some other triangles, trying to get the rhythm of the method and then forcing it into precise language. Even in this one example, if you stare long enough, you can see hidden meaning: the roles played by 13, 14, and 15 in the calculations, especially if you "delay the evaluation" and write the last equation as

$15^2 - 2 \cdot 15x + 13^2 = 14^2$

This delayed evaluation is one way to reveal hidden structure. And, if the triangle had side lengths a, b, and c, you'd end up with

$c^2 - 2cx + a^2 = b^2$

Here's where the structure of the expressions becomes dominant, and by careful transformations, one can use this to carry out a generic version of your numerical calculations and transform them into a derivation of Heron's formula, as this student did:



MORAL: It's impossible to put this approach into one of the buckets given by lists of standards. It employs in inextricable ways abstracting regularity, using precision, perseverance, and exploiting structure, all at once. Sometimes one habit becomes more dominant (as in the final transformations), but they are all here, all the time. And this is typical of how the practice of mathematics is exercised.

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