A New Vision for Developing Addition and Subtraction Computation Strategies

BY RANDALL I. CHARLES

When adults are asked to share a memory about their mathematics experiences in elementary school, many will remark about their challenges learning *standard algorithms* for adding, subtracting, multiplying, and dividing whole numbers. Table 1 shows calculations completed with standard algorithms. In this country, these algorithms are generally considered to be *the* standard algorithms for operations with whole numbers.

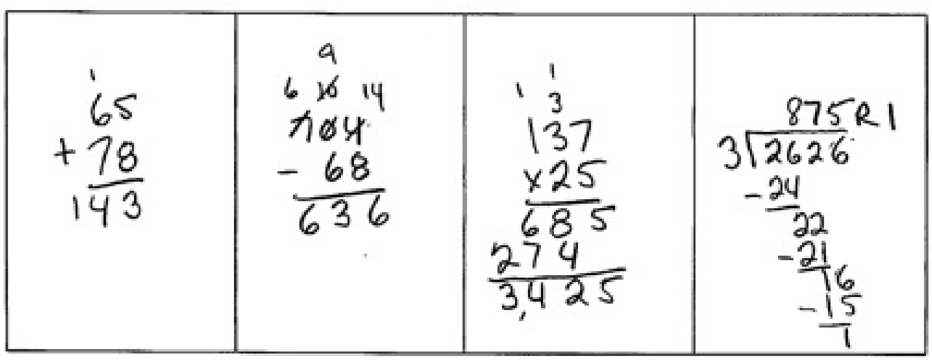
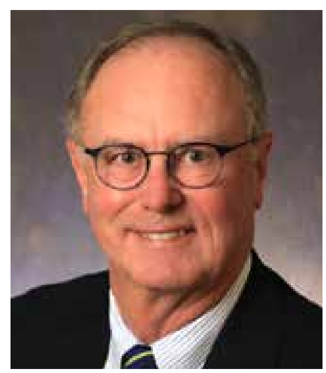


Table 1. Examples of the standard algorithms used in the United States for adding, subtracting, multiplying, and dividing whole numbers

Teaching standard algorithms has a long tradition in elementary mathematics education. As early as the 1800s, a majority of mathematics instructional time was devoted to developing facility with standard algorithms. Initial instruction was followed with extensive practice; then reteaching, which is now called *intervention*, followed as needed, along with more practice. Periodic review was common to help students maintain proficiency. Many adults today can recite the steps for some standard algorithms as easily as they can recite the alphabet—for example, when dividing with whole numbers, repeat the following steps: divide, multiply, subtract, compare, bring down!



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Until recently, standards for mathematics education have called for the elementary mathematics curriculum to be organized so that a standard algorithm is introduced at a particular grade with smaller numbers, extended to greater numbers at the next grade, extended to even greater numbers at the next grade, and so on, until a point where the process is generalized to numbers of essentially any size. Table 2 shows the traditional evolution of the standard algorithm for adding whole numbers.

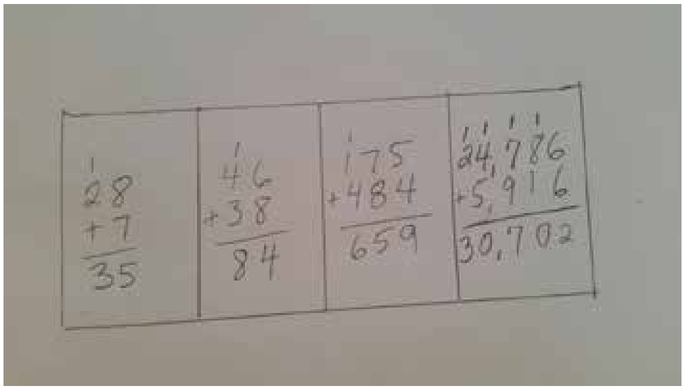


Table 2. Examples of the standard addition algorithm across the grades

Today, most state standards for mathematics present a different vision for developing computational facility adding, subtracting, multiplying, and dividing whole numbers. The essence of this vision is that prior to introducing a standard algorithm, instruction emphasizes *alternative computation strategies* that build on students' understanding of number, place value, properties, meanings of operations, and relationships among operations. Procedural or computational *fluency* is developed from a foundation of conceptual understanding.

The purpose of this paper is to describe and illustrate this vision for developing computational facility. Learning issues with a curriculum focused on standard algorithms will be discussed, followed by sample alternative computation strategies and their implications for learning and teaching. The discussion and samples in this paper are limited to strategies for adding and subtracting whole numbers because this different vision for computational facility can be explained and understood by focusing on these two operations.

A Different Vision for Curriculum Standards

Table 3 shows standards found in most state guidelines related to computational facility with addition and subtraction of multi-digit whole numbers.

Grade 1: Use place value understanding and properties of operations to add and subtract.
Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
Grade 2: Use place value understanding and properties of operations to add and subtract.
2.1 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
2.2 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.
2.2 Explain why addition and subtraction strategies work, using place value and the properties of operations.
Grade 3: Use place value understanding and properties of operations to perform multi-digit arithmetic.
Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
Grade 4: Use place value understanding and properties of operations to perform multi-digit arithmetic.
Fluently add and subtract multi-digit whole numbers using the standard algorithm. [Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.]

Source: National Governors Association Center for Best Practices and Council of Chief State School Officers (NGA Center and CCSSO, 2010).

Table 3. Fluency standards for Grades 1 through 4

Some important observations about these standards are the following:

- Alternative computation strategies are emphasized at Grades 1 through 3. The standard algorithms and fluency using them *are not introduced until Grade 4*.
- Limits are given for sums and related differences at each grade:

Grade 1: within 100
Grades 2 and 3: within 1,000
Grade 4: within 1,000,000

- Two phases of developing alternative computation strategies for adding and subtracting multi-digit whole numbers are made explicit:
  - Initial instruction should emphasize concrete models, drawings, and *strategies* based on place value, properties of operations, and/or the relationship between addition and subtraction.
  - Developing fluency should emphasize *strategies* based on place value, properties of operations, and/or the relationship between addition and subtraction.

The terms *fluency* or *fluently* in the standards for Grades 1 through 3 focus on students building strategies for addition and subtraction that engage them in decomposing and composing numbers in flexible ways. An appropriate way to think about fluency is “that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of operations to the eventual use of general methods as tools in solving problems” (NCTM 2014, p. 42).

However, teachers, as well as parents and administrators, often misinterpret *fluency* to mean that the curriculum should continue to promote mastery of the standard addition and subtraction algorithms at Grades 1 through 3, with sums and related differences up to certain limits. It is important that educators understand that fluency is not synonymous with mastery of standard algorithms.

This misinterpretation of fluency in Grades 1 through 3, along with standards calling for fluency before the introduction of standard algorithms, leads many teachers to rush through, or even omit, all work with alternative computational strategies and devote most or all of their instruction to the standard algorithms. As discussed later, introducing the standard addition and subtraction algorithms at Grades 1 through 3, even with the use of alternative computational strategies, is problematic. In fact, there is compelling evidence that the standard addition and subtraction algorithms for multi-digit whole numbers should be completely avoided at Grades 1 through 3.

The vision for developing computational facility reflected by the standards in Table 3 is not without controversy. Many parents and teachers expect and believe that one of the most important missions of elementary school mathematics is to develop in students the efficient and accurate use of standard algorithms for calculations with whole numbers, fractions, and decimals. Changes that challenge this tradition and expectation can make some teachers and parents uncomfortable. Beyond tradition, there are some beliefs that students need these specific skills to succeed on standardized tests and in their adult jobs. Parents want to support their child’s mathematics learning and many times feel most comfortable working with the mathematics they learned. The good news is that teachers and parents can still achieve these goals through this different vision for the development of computational facility.

WHAT ARE IMPORTANT ISSUES WITH A CURRICULUM FOCUSED ON STANDARD ALGORITHMS?

As mentioned earlier, the elementary school mathematics curriculum has for many years been dominated by teaching of the standard algorithms. In spite of the significant time spent developing, remediating, and reviewing these “skills,” data shows that too many students underperform in computation assessments (NCES, 2013).

For many years, students’ poor performance with computation was attributed to their lack of understanding of the standard algorithms; many educators believed that if students just understood the standard algorithms, they would perform better in the short term and retain these skills over time. As a result, major efforts and improvements were made in elementary mathematics curricula to teach the standard algorithms with understanding. Manipulative materials, pictorial representations, and an emphasis on place-value relationships became essential elements of instruction for developing standard algorithms with understanding. Also, extensive intervention resources were developed that embraced an emphasis on understanding. Unfortunately, “after the last 50 years or so of improvement in instruction and a host of well-intentioned approaches, we still have a lot of students making systematic errors." (Van de Walle, 2005). Research shows that too many students continue to build "buggy algorithms," or algorithms with errors that are often systematic and predictable. Even with instruction focused on understanding, many students have difficulty developing efficiency and accuracy with the standard algorithms. As a result, many of today's classrooms continue to spend an excessive amount of time teaching, practicing, and remediating standard algorithms.

Another important issue related to a curriculum focused on standard algorithms is that it promotes an incomplete view of what it means to "do mathematics". The following are some common beliefs about mathematics that a curriculum focused on standard algorithms promotes; these beliefs are untrue and counterproductive to students' immediate and future success with mathematics (Hiebert, 1984; Cobb, 1985; Baroody & Ginsburg, 1986).

- Mathematics consists mainly of symbols on paper.
- Following the rules for manipulating symbols on paper is of prime importance; mathematics is mostly memorization.
- Mathematics problems can be solved in less than 10 minutes; otherwise, they cannot be solved at all.
- Speed and accuracy are more important in mathematics than understanding.
- There is just one right way to solve any problem.
- Different, but correct, solution methods sometimes yield contradictory results.
- Mathematics symbols and rules have little to do with common sense, intuition, or the real world.

Students who acquire these beliefs about doing mathematics often build a dislike for mathematics. Fortunately, goals for mathematics education have now moved beyond ones connected to these beliefs.

Attributes of Alternative Computation Strategies

As mentioned above, *alternative computation strategies* are calculation methods that build on students' understanding of number, place value, properties, meanings of operations, and relationships among operations. Alternative computation strategies provide opportunities for students to capitalize on their mental math methods and be flexible as they compose (put together) and decompose (break apart) numbers in order to add, subtract, multiply or divide efficiently. In the process of reasoning through alternative computation strategies, teachers can naturally introduce properties of operations, such as the commutative, associative, and distributive properties, as students share their strategies.[1]

Table 4 shows significant differences between the standard algorithms and alternative computation strategies. These differences have a major impact on learning.

Standard Algorithms	Alternative Computation Strategies
• Standard algorithms are *digit oriented*. Steps in standard algorithms involve the accurate recall and use of basic facts with minor use of place-value understandings.	• Alternative computation strategies are *number oriented*. Digits are always connected to their place values.
• Standard algorithms are mostly *right-handed*. The standard algorithms for adding, subtracting, and multiplying whole numbers start at the right—they operate with digits in the ones place first, then the tens place, and so on.	• Alternative computation strategies are mostly *left-handed*. Because whole numbers are read and written from the left, and because alternative computational strategies are number oriented, alternative computational strategies usually begin with the greatest place value.
• Standard algorithms promote *rigidity*. The algorithm is used in cases where there are more efficient and accurate methods for getting answers.	• Alternative computation strategies promote *flexibility*. The operation and the nature of the numbers involved inform which calculation strategy is most efficient and accurate.
• Standards algorithms require *written* recordings of individual steps.	• Alternative computation strategies emphasize *mental math* with recordings used to support thinking.
• Standard algorithms provide a *single method* for each operation.	• Alternative computation strategies invite *multiple methods* to be used for the same calculation.
• Standard algorithms focus on *rules*. The actual steps in standard algorithms are generally not based on meaning.	• Alternative computation strategies focus on *number reasoning* that is based on place value relationships, meanings of operations, relationships among operations, and properties.

Source: National Governors Association Center for Best Practices and Council of Chief State School Officers (NGA Center and CCSSO, 2010).

Table 4. Differences between standard algorithms and alternative computation strategies

[1] See *Developing Essential Understandings of Addition and Subtraction: Pre-K–Grade.*

[2] (Caldwell et al, 2011) for a presentation of how properties underlie several of the sample strategies introduced in the next section.

Sample alternative computation strategies are shown and commented on in the next section. As you study each sample, think how each of the attributes of the alternative computational strategies shown in Table 4 plays out. In particular, think about other methods that might be used beyond the one given with each sample.

Sample Addition and Subtraction Strategies

It helps to understand addition and subtraction strategies by thinking about each strategy as one of two approaches (Fuson, 1990, 1992; Fuson, et al., 1997; Verschaffel, et al., 2007):

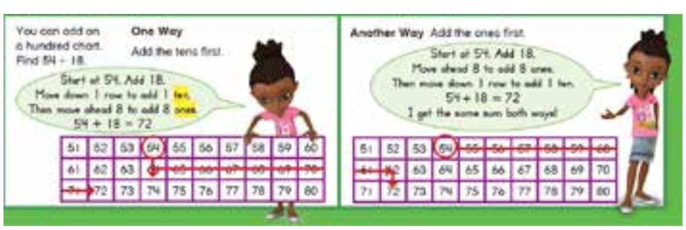
> **Approach 1:** Computations start with one number that is not decomposed, or broken apart.
>
> **Approach 2:** Computations start by decomposing both numbers into base-ten units.

However, this distinction is not important for students, and students' thinking about computation strategies often combines elements of these two approaches.

The following charts show sample strategies for adding and subtracting whole numbers. Addition and subtraction strategies related to Approach 1 are presented first followed by addition and subtraction strategies related to Approach 2. Some observations related to each strategy are given in the Comments sections.

ADDITION STRATEGIES: APPROACH 1—START WITH ONE NUMBER THAT IS NOT DECOMPOSED.

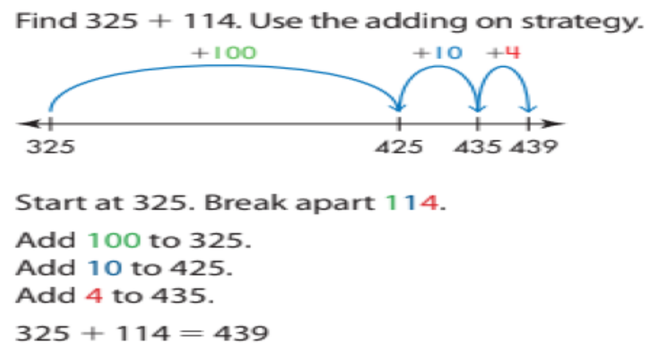
Add by Using a Hundred Chart



Source: **enVision**® Mathematics ©2020, Grade 2, Lesson 3-1, page 94 (Charles et al., 2020).

Comments: Addition using a hundred chart can be done by starting with either addend. However, it is usually easier to start with the greater number and then add on. Compensation is another method that is often used when adding on a hundred chart. For 54 + 18, students can count on 2 tens from 54 to 74 and then count back 2 ones from 74 to 72.

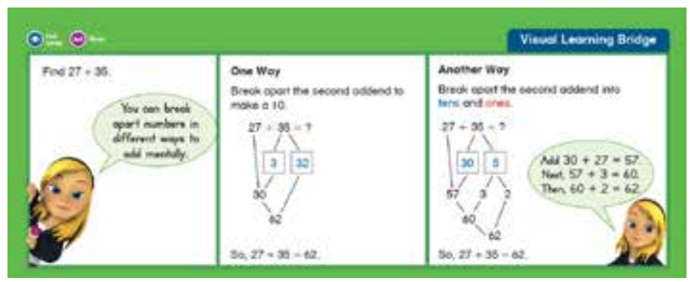
Add by Using an Open Number Line



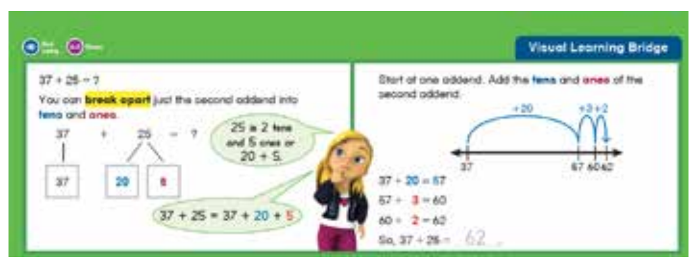
Source: **enVision**® Mathematics ©2020, Grade 3, Lesson 8-3, page 298 (Charles et al., 2020).

Comments: Similar to adding on a hundred chart, it is usually easier to start with the greater addend when adding on an open number line. The above example shows adding 100 first, but the place values can be added in any order.

Add by Breaking Apart One Addend



Source: **enVision**® Mathematics ©2020, *Grade 2, Lesson 4-5, page 154 (Charles et al., 2020).*

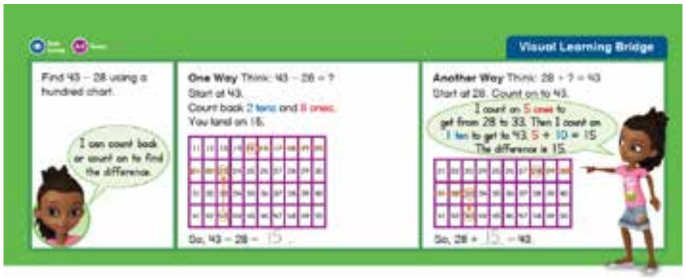


Source: **enVision**® Mathematics ©2020, *Grade 2, Lesson 3-3, p. 102 (Charles et al., 2020).*

Comments: In the top example (One Way), the second addend was decomposed into two numbers to make a ten. In the other examples, the second addend was broken into tens and ones.

SUBTRACTION STRATEGIES: APPROACH 1—START WITH ONE NUMBER THAT IS NOT DECOMPOSED.

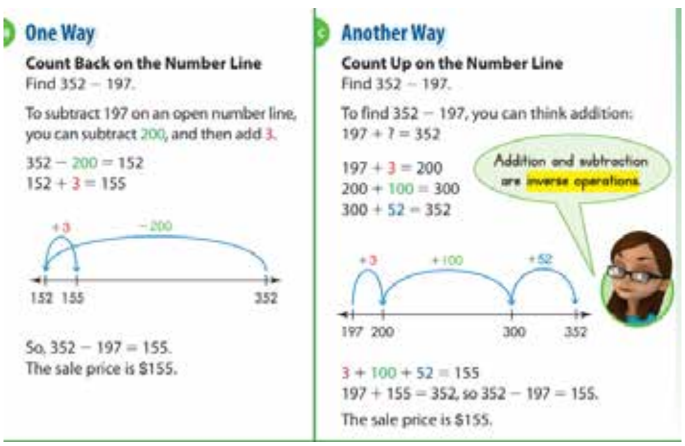
Subtract by Using a Hundred Chart



Source: **enVision®** Mathematics ©2020, *Grade 2, Lesson 5-1, p. 190 (Charles et al., 2020).*

Comments: Compensation is another common method for subtracting with a hundred chart. For this example, one can count back 3 tens from 43 to 13 and then count up 2 ones from 13 to 15.

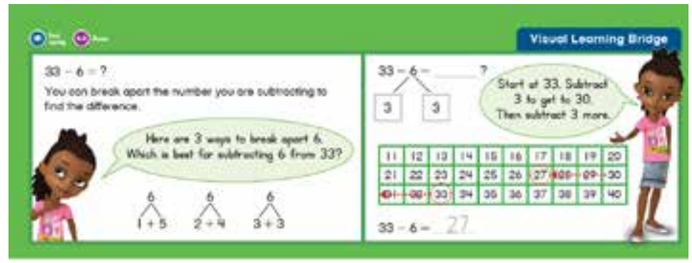
Subtract by Using an Open Number Line: Count Back or Count Up



Source: **enVision®** Mathematics ©2020, *Grade 3, Lesson 8-4, p. 302 (Charles et al., 2020).*

Comments: Compensation was used in the method shown at the left by first subtracting 200 from 352 and then adding 3 to the difference of 152. Another method is to count back by decomposing the subtrahend, 197, into hundreds, tens, and ones and then count back by each place value.

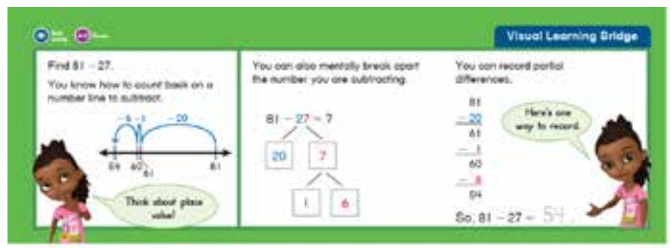
Subtract by Breaking Apart the Subtrahend in Order to Make the Minuend a Multiple of Ten



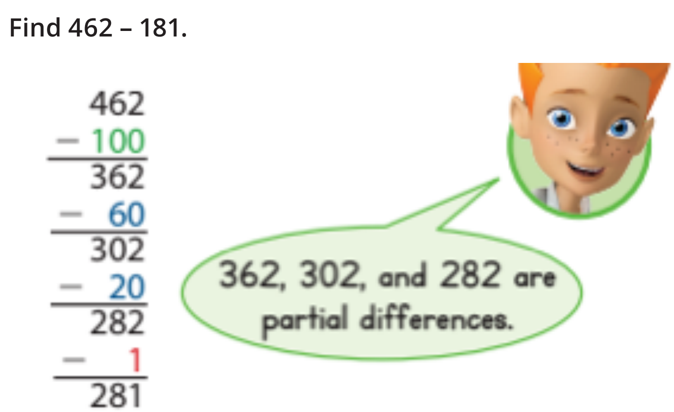
Source: **enVision®** Mathematics ©2020, *Grade 2, Lesson 5-4, p. 202 (Charles et al., 2020).*

Comments: This strategy is most efficient in problems beyond basic facts when a one-digit number is subtracted from a number with two or more digits.

Subtract by Finding a Partial Difference



Source: **enVision®** Mathematics ©2020,*Grade 2, Lesson 6-4, p. 250 (Charles et al., 2020).*

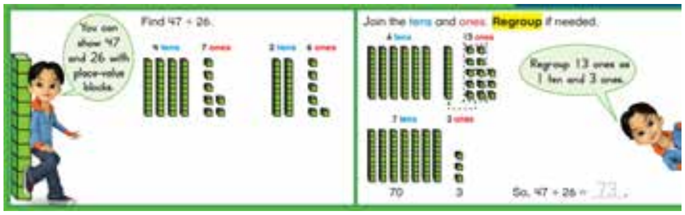


Source: **enVision®** Mathematics ©2020, *Grade 3, Lesson 9-4, p. 350 (Charles et al., 2020).*

Comments: The top example shows how partial differences are connected to counting back on the open number line, and how the subtrahend is broken apart using place value. When using partial differences, it is common to start subtracting in the greatest place value, but one can start by subtracting ones. In both examples, however, regrouping was not needed because 7 was broken apart into 1 and 6 and 80 was broken apart into 60 and 20.

ADDITION STRATEGIES: APPROACH 2—START BY DECOMPOSING BOTH NUMBERS INTO BASE-TEN UNITS.

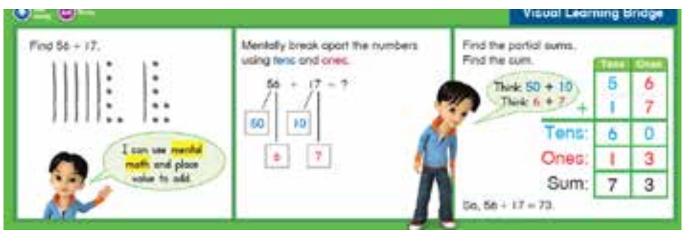
Add by Using Models



Source: **enVision®** Mathematics ©2020, *Grade 2, Lesson 4-1, p. 138 (Charles et al., 2020).*

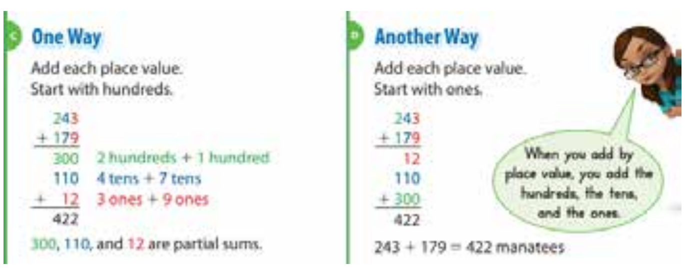
Comments: Both addends are decomposed into tens and ones and represented with base-ten materials. Using base-ten materials leads naturally to regrouping when models showing the same place value are joined.

Add by Using Partial Sums



Source: **enVision®** Mathematics ©2020, *Grade 2, Lesson 4-4, p. 150 (Charles et al., 2020).*

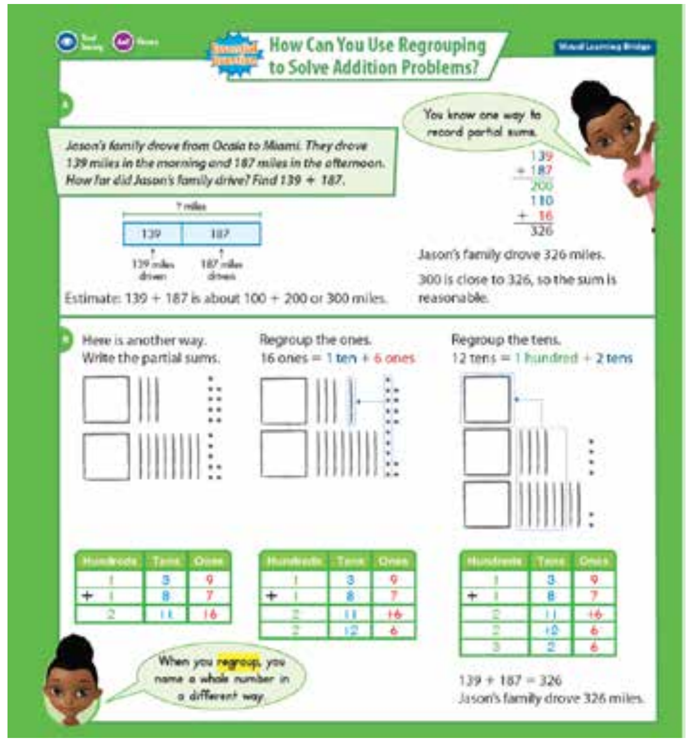
Find 243 + 179.



Source: **enVision®** Mathematics ©2020, *Grade 3, Lesson 8-X, p. X (Charles et al., 2020).*

Comments: Both addends are decomposed by their place values. The sequence in which the additions are completed does not matter when finding partial sums. In the above example at the right, tens could have been added first, then ones, and then hundreds. The answer would have been the same.

Add by Using Regrouping

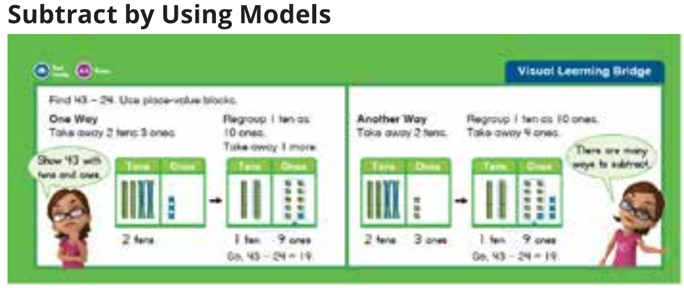


Source: **enVision®** Mathematics ©2020, *Grade 3, Lesson 9-2, p. 342 (Charles et al., 2020).*

Comments: An excellent transition to the concept of regrouping prior to introducing the standard algorithm can be seen by comparing the partial sums at the top of the page to the recording method at the bottom of the page.

SUBTRACTION STRATEGIES: APPROACH 2—START BY DECOMPOSING BOTH NUMBERS INTO BASE-TEN UNITS.

Subtract by Using Models

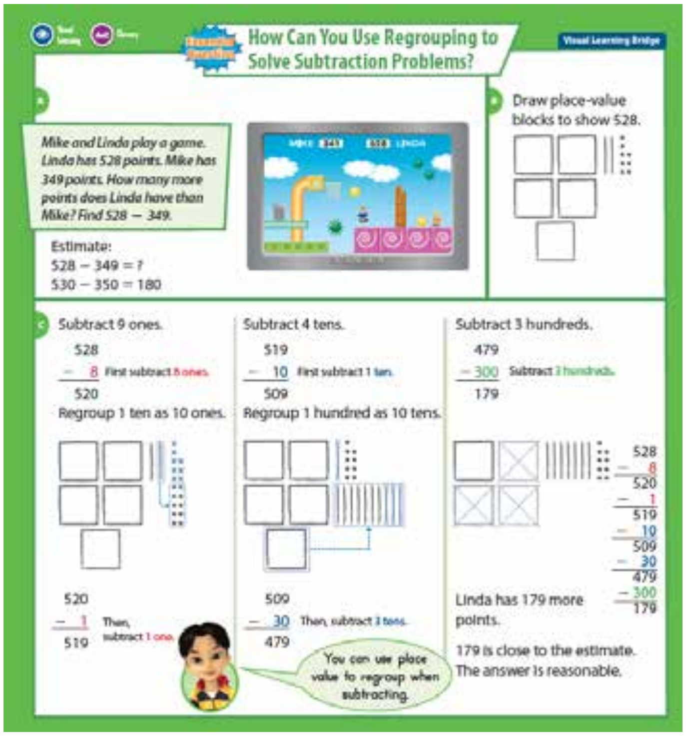


Source: **enVision®** Mathematics ©2020, *Grade 2, Lesson 6-2, p. 242 (Charles et al., 2020).*

Comments: The minuend is decomposed into tens and ones and represented with base-ten materials. Also, the use of base-ten materials requires students to regroup in order to have enough ones models to take away.

Subtract by Using Partial Differences: Regrouping

Find 528 – 349.



Source: **enVision®** Mathematics ©2020, *Grade 3, Lesson 9-5, p. 354 (Charles et al., 2020).*

Comments: This strategy combines parts of Approaches 1 and 2. Base-ten materials are used to decompose the minuend by place value. The subtrahend is mentally broken apart by place value in order to take away values in corresponding place values. However, the recording shows partial differences and looks like only one number is decomposed. Another way to say this is that the thinking used is Approach 2, but the recording used is Approach 1.

By comparing the thinking called for with these alternative strategies to the thinking called for with standard algorithms, it becomes clear that the thinking, reasoning, and sense making called for with alternative calculation strategies are profoundly different than the thinking involved with standard algorithms.

Some major benefits of a curriculum anchored on alternative calculation strategies include the following (Van De Walle et al., 2019, p. 248):

- Students make fewer errors.
- Less reteaching is required.
- Students develop number sense.
- Students become better at mental computation and estimation.
- Flexible methods are often faster than traditional.
- Students develop helpful attitudes and beliefs about what it means to *do* mathematics.

Important Guidelines

Listed below are some important guidelines for shifting the curriculum to a focus on alternative calculation strategies:

- The main focus should be on reasoning and sense making. These strategies should not be presented as alternative *algorithms* to be mastered. Just as it is possible to teach the standard algorithms as a sequence of rote steps, alternative computation strategies also can be presented this way. However, this should be avoided.
- "Students should be encouraged to use whatever methods they wish and to use only methods that they understand and can explain." (Van de Walle, 2005) The goal is not for *all* students to become proficient with *each* alternative computation strategy the teacher shares or students invent. Also, a student should not be required to use a particular strategy when he or she prefers a different strategy that is both efficient and accurate.
- Students should be encouraged to choose strategies flexibly and to use different strategies based on the numbers in the calculation.
- Teachers should encourage and be receptive to other student-invented calculation strategies. "Student-created strategies can and should be shared by students and adopted by others when and if they see how the methods make sense. This means students should be encouraged to 'borrow' strategies from their peers." (Van de Walle, 2005)[2]

[2]Some of the ways to encourage these strategies is through "number talks" (Humphreys & Parker, 2015; Parrish, 2014) in which students share alternate strategies for computation. By listening and learning from their classmates' alternate and invented strategies, students grow their own repertoire of strategies.

Connecting Alternative Computation Strategies and Standard Algorithms

When the benefits from using alternative calculation strategies became known, curricula tried to do it all; that is, introduce alternative calculation strategies followed by work with the standard algorithm at *each grade*. However, research has shown that the benefits previously given from working with alternative computation strategies are lost when instruction at each grade starts with alternative computation strategies and then shifts to the standard algorithm (Carpenter et al, 1998). Once a standard algorithm is introduced, research shows the difficulty of returning to other strategies; students resist returning to reasoning and trying to make sense of other strategies. "One hypothesis [for why it is difficult to return to sense-making strategies] is ... that the traditional algorithms, when compared to ... strategies, require much less cognitive effort. Students need only to utilize their knowledge of facts and follow the rules. A second hypothesis is that students see the traditional algorithms as those used by adults and that come from adults or others they respect."
(Van de Walle, 2005)

Most standards now call for introduction of the standard algorithms for addition and subtraction at Grade 4.[3] At this grade, the standard algorithms can be presented as natural extensions of and connections to several strategies students have used for three years. In particular, the strategies for Approach 2 shown previously are closely connected to the processes involved in the standard addition and subtraction algorithms.

Challenges to Attaining the Vision of the Standards

Some of the challenges to be confronted in shifting the curriculum toward the vision of the standards shown in Table 3 have already been mentioned throughout this paper. Some of the main reasons that this shift may be challenging are:

- Teachers' education did not include work with alternative calculation strategies. So, many teachers do not have the reasoning and sense-making inclinations and skills needed to support students. Teacher education is a key component to the success of this curriculum shift.
- Parents' mathematics education did not include work with alternative calculation strategies. Many parents do not see the value of emphasizing strategies in the early grades and many certainly will be concerned about delaying the introduction of standard algorithms. Parents become concerned when they feel unprepared to help their children with mathematics. Parent education is another key component to the success of this curriculum shift.
- Many curriculum materials have not fully embraced the vision of recent standards for developing computational fluency. Most curricula include alternative calculation strategies to some extent, but then return to "borrowing" and "carrying" and the standard ways of recording addition and subtraction calculations *at each grade*. A different vision for computational fluency will be attained only when teachers have access to curriculum materials that are *fully aligned* to the standards.

Conclusion

There is sufficient evidence showing that the standard addition and subtraction algorithms *should not be included* in Grades 1 through 3. New standards aligned with those in Table 3 must be taken literally; the vision for developing computational fluency embraced by these standards cannot be partially pursued.

Moving toward the vision for computational fluency embraced by recent curriculum standards means taking on a tradition that has been in place for over a hundred years. There are obstacles and challenges to attaining a different vision, but the results are worth the effort. A curriculum that develops computational fluency as outlined in this paper will enable more students to become proficient in calculating, will improve students' reasoning abilities, will help students succeed in future mathematics, and will convince all students and teachers that mathematics makes sense.

[3]The standard algorithm for multiplying multi-digit whole numbers is introduced in recent standards at Grade 5, and the standard algorithm for dividing multi-digit whole numbers is at Grade 6.

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